Finite energy chiral sum rules in QCD[a]

C. A. Dominguez
Institute of Theoretical Physics and Astrophysics
University of Cape Town, Rondebosch 7700, South Africa

K. Schilcher
Institut für Physik, Johannes Gutenberg-Universität
Staudingerweg 7, D-55099 Mainz, Germany

Abstract

A set of well known chiral sum rules, expected to be valid in QCD, is confronted with experimental data on the vector and axial-vector hadronic spectral functions, obtained from tau-lepton decay by the ALEPH collaboration. The Das-Mathur-Okubo sum rule, the first and second Weinberg sum rules, and the electromagnetic pion mass difference sum rule are not well saturated by the data. Instead, a modified set of sum rules having additional weight factors that vanish at the end of the integration range on the real axis, is found to be precociously saturated by the data to a remarkable extent.

There is a set of sum rules, first discovered in the framework of current algebra, which are now understood as consequences of the underlying chiral symmetry of QCD. We refer to the Das-Mathur-Okubo sum rule (DMO) [1]

$$W_0 \equiv \int_0^\infty \frac{ds}{s}\,[\rho_V(s) - \rho_A(s)] = \frac{1}{3} f_\pi^2 < r_\pi^2 > - F_A$$

$$= -4\bar{L}_{10} = (2.73 \pm 0.12) \times 10^{-2} \;, \qquad (1)$$

the first and second Weinberg sum rules [2]

$$W_1 \equiv \int_0^\infty ds\;[\rho_V(s) - \rho_A(s)] = f_\pi^2 \;, \qquad (2)$$

$$W_2 \equiv \int_0^\infty ds\; s\;[\rho_V(s) - \rho_A(s)] = 0 \;, \qquad (3)$$

and the electromagnetic pion mass difference sum rule [3]

$$W_3 \equiv \int_0^\infty ds\; s \ln\left(\frac{s}{\mu^2}\right)\,[\rho_V(s) - \rho_A(s)] = -\frac{16\,\pi^2\,f_\pi^2}{3\,e^2}\,\left(\mu^2_{\pi^\pm} - \mu^2_{\pi^0}\right) \;. \qquad (4)$$

[a] Work supported in part by the Volkswagen Foundation

In the above equations $\rho_{V,A}(s)$ are, respectively, the imaginary parts of the vector and axial-vector two-point functions

$$\Pi_{\mu\nu}^{VV}(q^2) = i\int d^4x e^{iqx} < 0|T(V_\mu(x)V_\nu^\dagger(0))|0 >= (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi_V(q^2) \; , \tag{5}$$

$$\begin{aligned}\Pi_{\mu\nu}^{AA}(q^2) = i \int \; d^4\, x\, e^{iqx} \quad < 0|T(A_\mu(x) \;\; A_\nu^\dagger(0))|0 > \\ = \; (-g_{\mu\nu}\, q^2 + q_\mu q_\nu)\, \Pi_A(q^2) - q_\mu q_\nu\, \Pi_0(q^2) \; ,\end{aligned} \tag{6}$$

where $V_\mu(x) =: \bar{q}(x)\gamma_\mu q(x) :$, $A_\mu(x) =: \bar{q}(x)\gamma_\mu\gamma_5 q(x) :$, with $q = (u, d)$, the pion decay constant is $f_\pi = 92.4 \pm 0.26$ MeV [4], the pion electromagnetic mean square radius has the value $< r_\pi^2 >= 0.439 \pm 0.008$ fm^2 [5], and F_A is the axial-vector coupling measured in radiative pion decay, $F_A = 0.0058 \pm 0.0008$ [4]. The above set of sum rules are exact relations in QCD, in the chiral $SU(2)_L \times SU(2)_R$ limit, i.e. for vanishing up- and down-quark masses [6]. At short distances, the spectral functions are strictly identical in the chiral limit ($m_q = 0$), to all orders in QCD perturbation theory, and up to dimension-four non-perturbatively. The difference between ρ_V and ρ_A, always at short distances, is non-perturbative and starts at dimension-six involving the four-quark condensate. On the other hand, the low energy behaviour of $\Pi_{V-A}(q^2)$ is governed by chiral perturbation theory. In confronting these sum rules with experimental data on the spectral functions, one must bear in mind that the latter are known up to a certain finite energy $\sqrt{s_0}$; in the case of the ALEPH data [7] this is $s_0 < M_\tau^2 < \infty$. Invoking quark-hadron duality, the sum rules Eqs. (1)-(4) become effectively Finite Energy Sum Rules (FESR). The question is then, how well is local and global duality satisfied by the experimental data. It has been argued recently [8] that the agreement between data and theory, which measures the validity of local duality, is not entirely satisfactory. In view of this, one would expect a similar unsatisfactory saturation of the four chiral sum rules, Eqs.(1)-(4), and in fact, this is what we find. Before illustrating our results, let us write down a set of four chiral sum rules modified by an integral kernel vanishing at $s = s_0$ on the real axis. This modification may be referred to as *restricted global duality*, which is expected to set in much sooner than ordinary global duality. The sum rules are

$$\bar{W}_0 \;\equiv\; \int_0^\infty \frac{ds}{s}\,(1 - \frac{s}{s_0})\;\;[\rho_V(s) - \rho_A(s)] = -4\,\bar{L}_{10} \;-\; \frac{f_\pi^2}{s_0} \; , \tag{7}$$

$$\bar{W}_1 \equiv \int_0^\infty \; ds\,(1 - \frac{s}{s_0})\;\;[\rho_V(s) - \rho_A(s)] = f_\pi^2 \; , \tag{8}$$

$$\bar{W}_2 \equiv \int_0^\infty \frac{ds}{s}\,(1-\frac{s}{s_0})^2\;[\rho_V(s)-\rho_A(s)] \;=\; -4\bar{L}_{10} - 2\frac{f_\pi^2}{s_0}\;, \tag{9}$$

$$\bar{W}_3 \equiv \int_0^\infty ds\; s\,\ln\left(\frac{s}{s_0}\right)\;[\rho_V(s)-\rho_A(s)] = -\frac{16\;\pi^2\;f_\pi^2}{3\;e^2}\;\left(\mu^2_{\pi^\pm}\;-\mu^2_{\pi^0}\right)\;, \tag{10}$$

where Eq.(7) is a combination of the DMO sum rule and the first Weinberg sum rule, Eq.(9) is a combination of the DMO sum rule and the first and second Weinberg sum rules, and the arbitrary scale in Eq.(10) has been fixed to s_0, which becomes the upper limit of integration in the four sum rules. We now show that these modified sum rules are saturated far better than the original sum rules Eqs.(1)-(4). In Fig.1 we plot the left hand side (l.h.s.) of Eq.(1) computed using the fit to the data (curve(a)), and the right hand side (r.h.s.) (curve(b)). Agreement with the data can be considerably improved by rescaling the r.h.s. of Eq.(1) from the value 2.73×10^{-2} to 2.43×10^{-2}. Figure 2 shows the l.h.s. of the modified DMO sum rule Eq.(7) (curve(a)) compared to the r.h.s. (curve (b)) after performing the above rescaling. In Fig.3 we plot the l.h.s. of the first Weinberg sum rule Eq.(2) (curve(a)), its modified version, Eq.(8) (curve(b)), and their r.h.s. (curve(c)). Figure 4 shows the l.h.s. of the second Weinberg sum rule Eq.(3) (curve(a)), compared to its r.h.s. (curve(b)), and Fig. 5 the corresponding curves for the modified sum rule, Eq.(9). Finally, in Fig. 6 we plot the l.h.s. of the sum rule Eq.(4) (curve(a)), the l.h.s. of the modified sum rule, Eq.(10) (curve (b)), and their r.h.s. (curve(c)). An inspection of these results clearly indicates that, the original chiral sum rules do not appear well saturated by the data. While an overall constant rescaling of the experimental data can result in a better saturation of the DMO sum rule, this would not help with the other three sum rules. Hence, the problem cannot be blamed on a systematic overall normalization uncertainty in the data. On the other hand, by using *restricted global duality*, the four modified chiral sum rules Eqs.(7)-(10) are extremely well saturated by the data.

1. T. Das, V.S. Mathur, and S. Okubo, Phys. Rev. Lett. **19** (1967) 859.
2. S. Weinberg, Phys. Rev. Lett. **18** (1967) 507.
3. T. Das, G.S. Guralnik, V.S. Mathur, F.E. Low, and J.E. Young, Phys. Rev. Lett. **18** (1967) 759.
4. Particle Data Group, R.M. Barnett et al., Phys. Rev. **D54** (1996) 1.
5. S.R. Amendolia et al., Nucl. Phys. **B277** (1986) 168.
6. E.G.Floratos, S.Narison, and E.de Rafael, Nucl.Phys.**B155** (1979) 115.
7. ALEPH Collaboration, R. Barate et al., CERN Report No. CERN-EP/98-12 (1998).
8. C.A. Dominguez and K. Schilcher, Phys. Lett. **B448** (1999) 93.

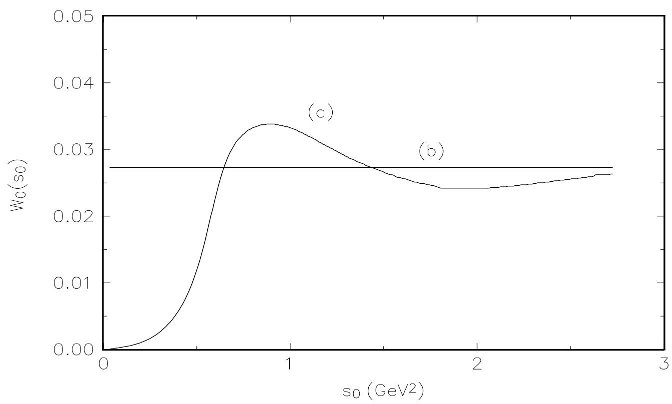

Figure 1:

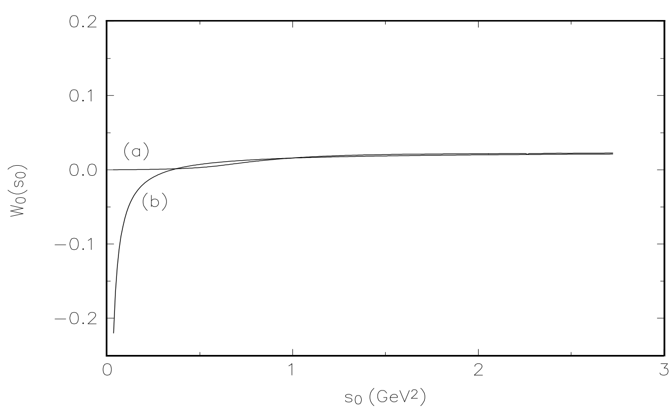

Figure 2:

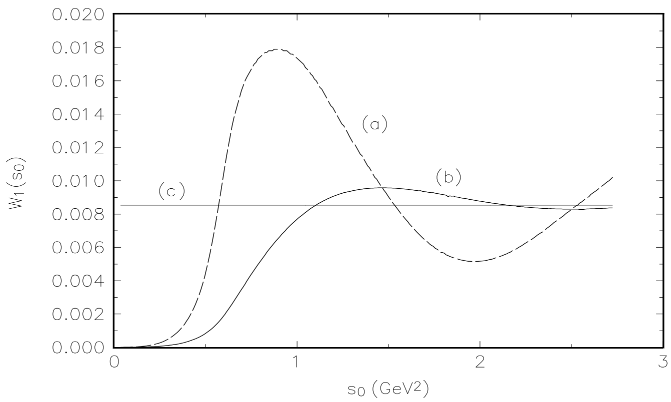

Figure 3:

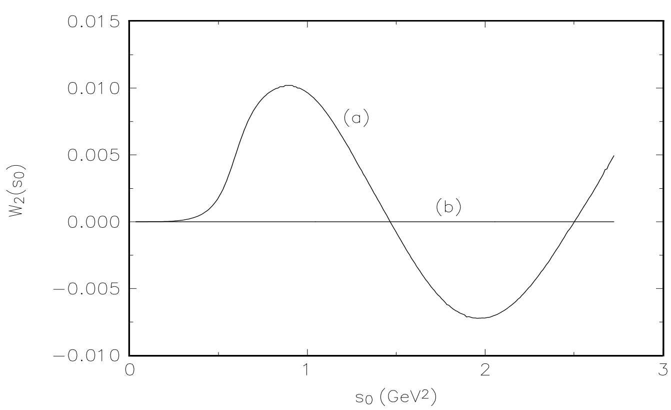

Figure 4:

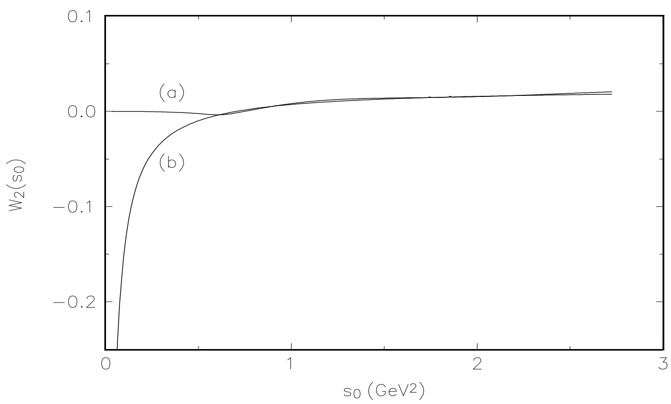

Figure 5:

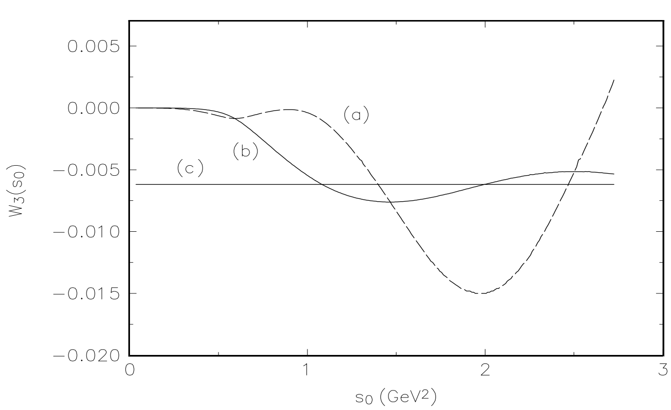

Figure 6: